Regular Polytopes

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Outline

How many regular polytopes are there in n dimensions?

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- Definitions and examples
- Platonic solids
  - Why only five?
  - How to describe them?
- Regular polytopes in 4 dimensions
- Regular polytopes in higher dimensions

Polytope is the general term of the sequence
"point, segment, polygon, polyhedron,. . . "

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Definition

A polytope in $\mathbb{R}^n$ is a finite, convex region enclosed by a finite number of hyperplanes. We denote it by Π_n.

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A polytope in $\mathbb{R}^n$ is a finite, convex region enclosed by a finite number of hyperplanes. We denote it by Π_n.

Examples $n = 0, 1, 2, 3, 4$.

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1. regular facets
2. regular vertex figures

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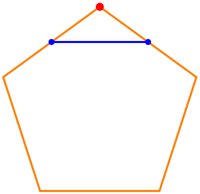
Vertex figure at vertex v is a Π_{n-1} obtained by joining the midpoints of adjacent edges incident to v.



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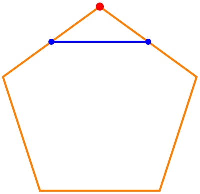
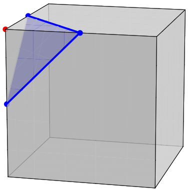
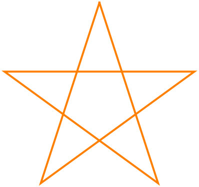
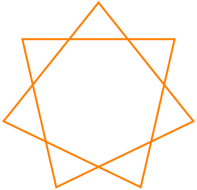




Star-polygons



$\left\{\frac{5}{2}\right\}$



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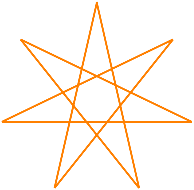

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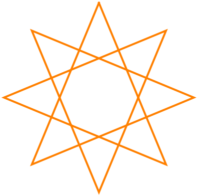
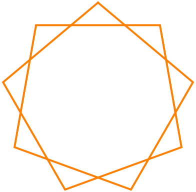
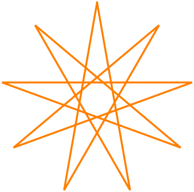


$\left\{\frac{8}{3}\right\}$

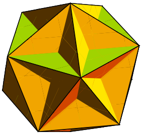


$\left\{\frac{9}{2}\right\}$

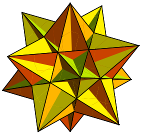


$\left\{\frac{9}{4}\right\}$

Kepler-Poinsot solids



$\{5, \frac{5}{2}\}$



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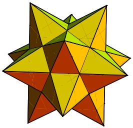
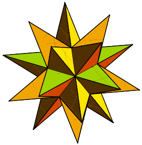


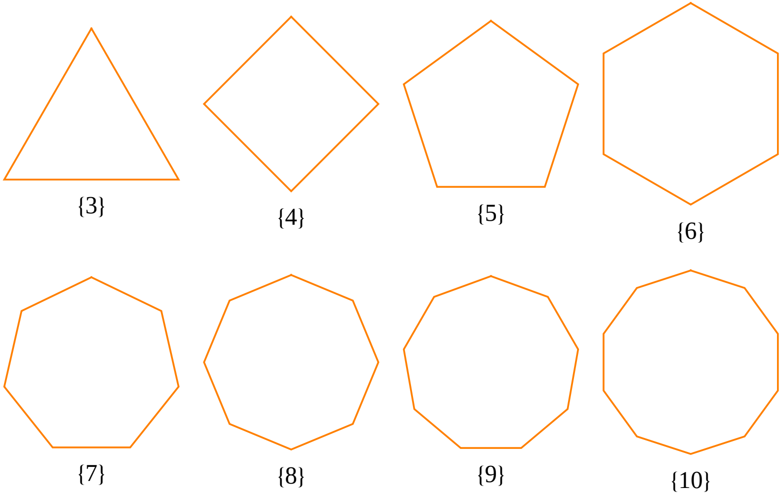
$\{\frac{5}{2}, 5\}$



$\{\frac{5}{2}, 3\}$

Two dimensional case

In 2 dimensions there is an infinite number of regular polytopes (polygons).



Necessary condition in 3D

Polyhedron $\{p, q\}$

- Faces of polyhedron are polygons $\{p\}$
- Vertex figures are polygons $\{q\}$. Note that this means that exactly q faces meet at each vertex.

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$$\frac{1}{2} < \frac{1}{p} + \frac{1}{q}$$

Solutions of the inequality

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$p = 3$	$p = 4$	$p = 5$

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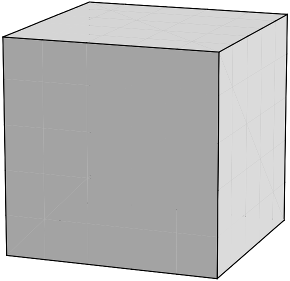
$q = 3, 4, 5$	$q = 3$	$q = 3$

But do the corresponding polyhedrons really exist?

$$\{p, q\} = \{4, 3\}$$

Cube

$$\{p, q\} = \{4, 3\}$$

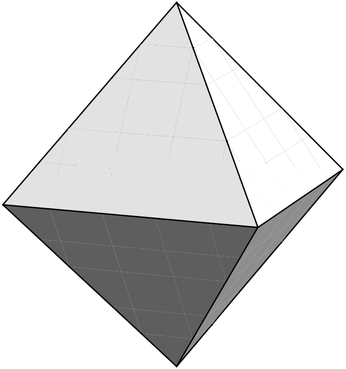


$$(\pm 1, \pm 1, \pm 1)$$

$$\{p, q\} = \{3, 4\}$$

Octahedron

$$\{p, q\} = \{3, 4\}$$



$$(\pm 1, 0, 0)$$
$$(0, \pm 1, 0)$$
$$(0, 0, \pm 1)$$

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Tetrahedron

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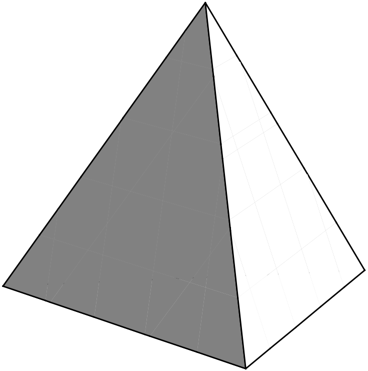

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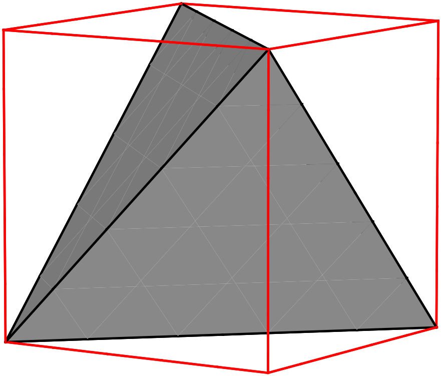

$(+1, +1, +1)$
$(+1, -1, -1)$
$(-1, +1, -1)$
$(-1, -1, +1)$

$$\{p, q\} = \{3, 5\}$$

Icosahedron

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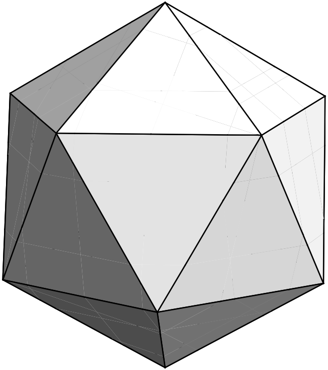

Icosahedron

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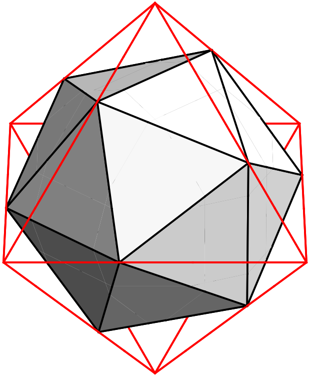

$$(0, \pm\tau, \pm 1)$$
$$(\pm 1, 0, \pm\tau)$$
$$(\pm\tau, \pm 1, 0)$$

where

$$\tau = \frac{1 + \sqrt{5}}{2}$$

$$\{p, q\} = \{5, 3\}$$

Dodecahedron

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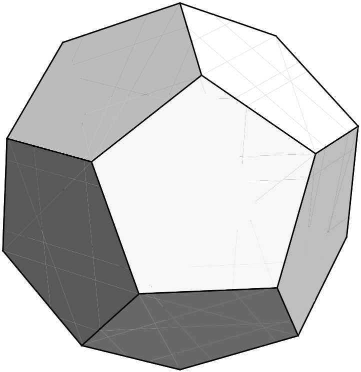

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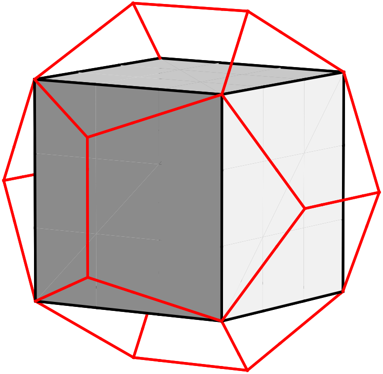

$$(\pm 1, \pm 1, \pm 1)$$
$$(0, \pm\tau, \pm\tfrac{1}{\tau})$$
$$(\pm\tfrac{1}{\tau}, 0, \pm\tau)$$
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where

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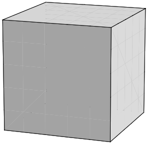
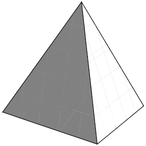
Five Platonic solids



Cube

{4, 3}



Tetrahedron

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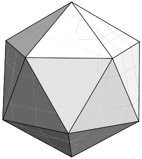
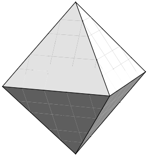


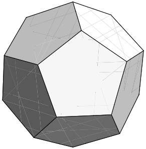
Icosahedron

{3, 5}



Octahedron

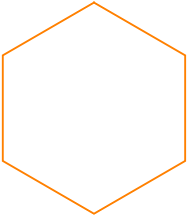
{3, 4}



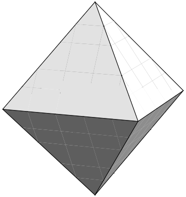
Dodecahedron

{5, 3}

Schläfli symbol



{6}



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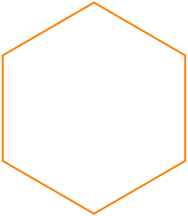

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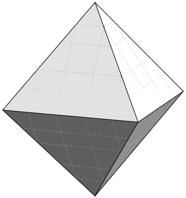

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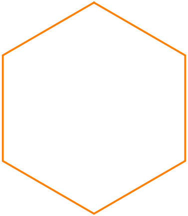
Desired properties of a Schläfli symbol of a regular polytope Π_n

1. Schläfli symbol is an ordered set of $n - 1$ natural numbers

Schläfli symbol



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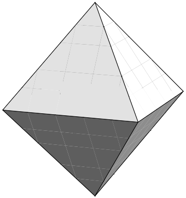

{3, 4}

Desired properties of a Schläfli symbol of a regular polytope Π_n

1. Schläfli symbol is an ordered set of $n - 1$ natural numbers
2. If Π_n has Schläfli symbol $\{k_1, k_2 \ldots, k_{n-1}\}$, then its
   - Facets have Schläfli symbol $\{k_1, k_2 \ldots, k_{n-2}\}$.
   - Vertex figures have Schläfli symbol $\{k_2, k_3 \ldots, k_{n-1}\}$.

Schläfli symbol

Claim

Vertex figure of a facet is a facet of a vertex figure.

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If Π_4 is a regular polytope, then it has

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We define the Schläfli symbol of Π_4 to be $\{p, q, r\}$.

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In general if Π_n is a regular polytope, then it has

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Thus the Schläfli symbol of Π_n is $\{k_1, k_2, \ldots, k_{n-1}\}$.

Regular 4-dimensional polytopes

Regular polyhedrons

$\{3,3\}, \{3,4\}, \{3,5\}, \{4,3\}, \{5,3\}$

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By superimposing we can form the following Schläfli symbols:

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Regular 4-dimensional polytopes

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$$\cancel{\{3,5,3\}}$$
$$\{4,3,3\},{\color{red}\{4,3,4\}},\cancel{\{4,3,5\}}$$
$$\{5,3,3\},\cancel{\{5,3,4\}},\cancel{\{5,3,5\}}$$

Regular 5-dimensional polytopes

Six regular 4-dimensional polytopes

$\{3,3,3\}, \{3,3,4\}, \{3,3,5\}, \{3,4,3\}, \{4,3,3\}, \{5,3,3\}$

Regular 5-dimensional polytopes

Six regular 4-dimensional polytopes

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$$\{4,3,3,3\}, \{4,3,3,4\}, \{4,3,3,5\}$$

Regular 5-dimensional polytopes

Six regular 4-dimensional polytopes

$\{3,3,3\}, \{3,3,4\}, \{3,3,5\}, \{3,4,3\}, \{4,3,3\}, \{5,3,3\}$

By superimposing we can form the following Schläfli symbols:

$$\{3,3,3,3\}, \{3,3,3,4\}, \{3,3,3,5\}$$
$$\{3,3,4,3\}$$
$$\{3,4,3,3\}$$
$$\{4,3,3,3\}, \{4,3,3,4\}, \{4,3,3,5\}$$
$$\{5,3,3,3\}, \{5,3,3,4\}, \{5,3,3,5\}$$

Regular 5-dimensional polytopes

Six regular 4-dimensional polytopes

$\{3,3,3\}, \{3,3,4\}, \{3,3,5\}, \{3,4,3\}, \{4,3,3\}, \{5,3,3\}$

By superimposing we can form the following Schläfli symbols:

$$\{3,3,3,3\}, \{3,3,3,4\}, \sout{\{3,3,3,5\}}$$
$$\{3,3,4,3\}$$
$$\{3,4,3,3\}$$
$$\{4,3,3,3\}, \{4,3,3,4\}, \sout{\{4,3,3,5\}}$$
$$\sout{\{5,3,3,3\}}, \sout{\{5,3,3,4\}}, \sout{\{5,3,3,5\}}$$

Regular 5-dimensional polytopes

Six regular 4-dimensional polytopes

$\{3,3,3\}, \{3,3,4\}, \{3,3,5\}, \{3,4,3\}, \{4,3,3\}, \{5,3,3\}$

By superimposing we can form the following Schläfli symbols:

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$\{3,3,4,3\}$

$\{3,4,3,3\}$

$\{4,3,3,3\}, \{4,3,3,4\},$ ~~$\{4,3,3,5\}$~~

~~$\{5,3,3,3\}$~~, ~~$\{5,3,3,4\}$~~, ~~$\{5,3,3,5\}$~~

Three regular 5-dimensional polytopes

$\{3,3,3,3\}, \{3,3,3,4\}, \{4,3,3,3\}$

Three regular 5-dimensional polytopes

$\{3,3,3,3\}, \{3,3,3,4\}, \{4,3,3,3\}$

Proceeding in the same manner we can form the following Schläfli symbols:

$$\alpha_n = \{3,3,\ldots,3,3\} = \{3^{n-1}\} \text{ Simplex}$$
$$\beta_n = \{3,3,\ldots,3,4\} = \{3^{n-2},4\} \text{ Cross polytope}$$
$$\gamma_n = \{4,3,\ldots,3,3\} = \{4,3^{n-2}\} \text{ Hypercube}$$

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We can also get $\{4,3,\ldots,3,4\} = \{4,3^{n-3},4\}$, but it turns out to be a honeycomb.

Summary

Dimension	1	2	3	4	≥ 5
Number of polytopes	1	∞	5	6	3